Introduction of Gaussian Beam
- Derivation of Laguerre-Gaussian mode -

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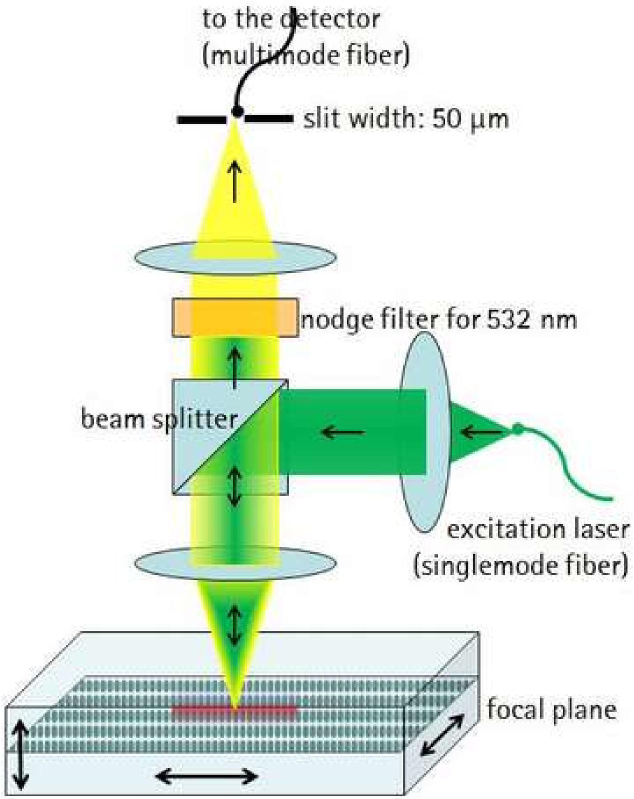
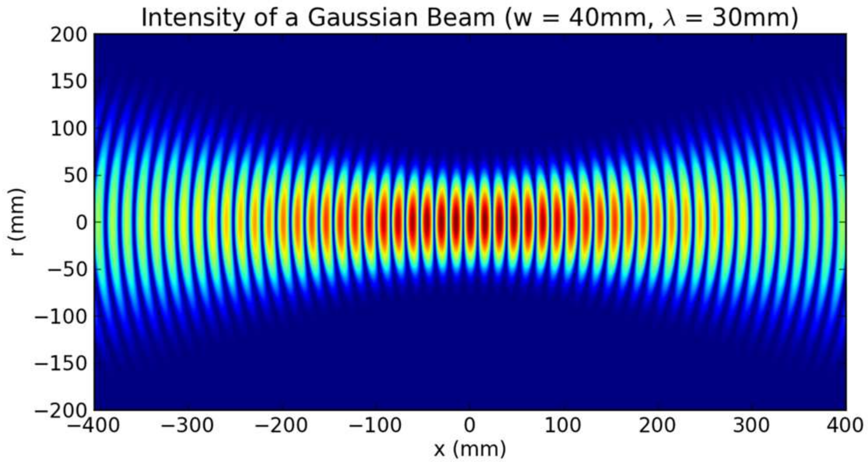


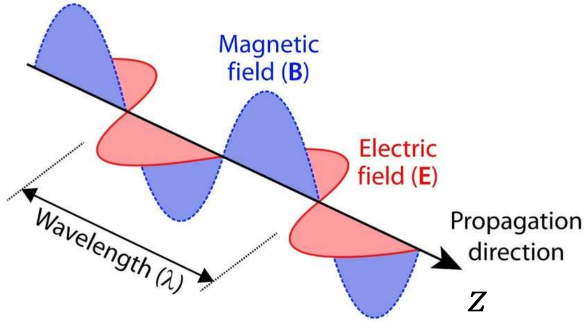
Confocal Raman spectroscopy



Paraxial Helmholtz Equation

Wave Eq. for Electric field (we can obtain from the Maxwell Eq.)

$$\Delta \boldsymbol{E} = \frac{1}{c^2}\frac{\partial^2 \boldsymbol{E}}{\partial t^2}$$



When $E \,//\, u \propto e^{-i\omega t}$, we get the Helmholtz Eq.

$$\Delta u + k^2 u = 0 \quad (k = \omega / c)$$

Propagating in the direction of z: $\quad u = f(x,y,z)e^{ikz}$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} + 2ik\frac{\partial f}{\partial z} = 0$$

Further if f changes slowly as a function of z compared with the wavelength

$$\frac{\partial^2 f}{\partial z^2} << k\frac{\partial f}{\partial z} \rightarrow \boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + 2ik\frac{\partial f}{\partial z} = 0 \quad \text{(Paraxial Helmholtz Equation)}}$$

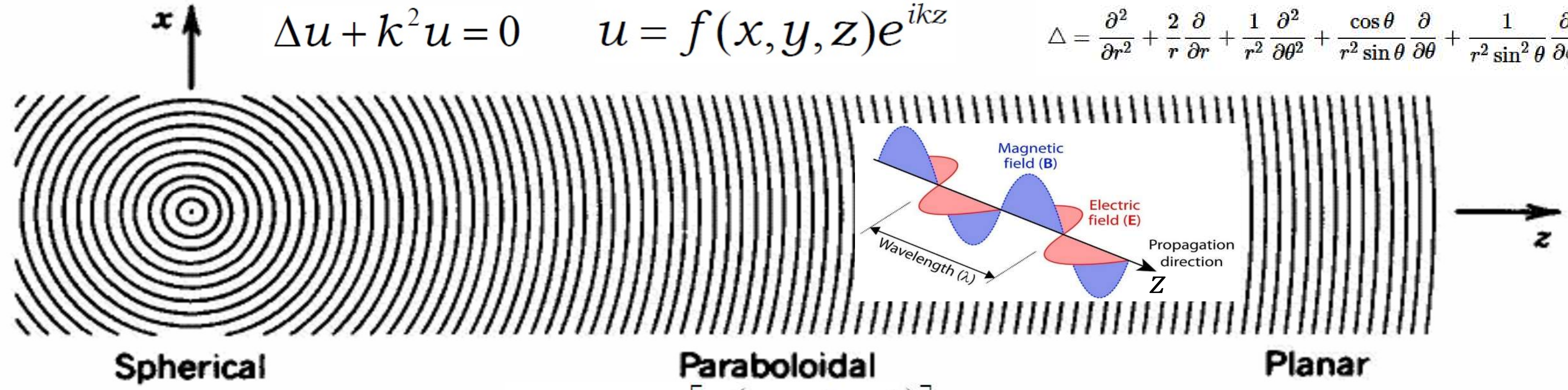
$$\kappa = \frac{2\pi}{\lambda}$$

Solutions of Paraxial Helmholtz Eq.

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+2ik\frac{\partial f}{\partial z}=0 \qquad \text{(Paraxial Helmholtz Eq.)}$$

$$\Delta u + k^2 u = 0 \qquad u = f(x,y,z)e^{ikz}$$

$$\triangle = \frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\cos\theta}{r^2\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}$$



$$u(\boldsymbol{r}) = \frac{A}{r}e^{ikr} \longrightarrow u(\boldsymbol{r}) = \frac{A}{z}\exp\left[ik\left(z+\frac{x^2+y^2}{2z}\right)\right] \qquad u(\boldsymbol{r}) = Ae^{ikz}$$

$$\left(f = \frac{A}{z}\exp\left[ik\frac{x^2+y^2}{2z}\right]\right) \qquad (f=1)$$

$$\frac{\partial u}{\partial r} = \left(i\kappa - \frac{1}{r}\right)u$$

$$\frac{\partial^2 u}{\partial r^2} = \left(i\kappa - \frac{1}{r}\right)^2 u + \frac{u}{r^2} = \left(-\kappa^2 - \frac{2i\kappa}{r} + \frac{2}{r^2}\right)u$$

$$\therefore \left(\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\kappa^2\right)u = 0$$

$$r = \left(x^2+y^2+z^2\right)^{1/2} = z\left(1+\frac{x^2+y^2}{z^2}\right)^{1/2} \cong z + \frac{x^2+y^2}{2z}$$

$$\Delta u + k^2 u = 0$$

$$\frac{\partial f}{\partial x} = f\cdot\left(\frac{i\kappa x}{z}\right)$$

$$\frac{\partial^2 f}{\partial x^2} = f\cdot\left(\frac{i\kappa x}{z}\right)^2 + f\cdot\frac{i\kappa}{z}$$

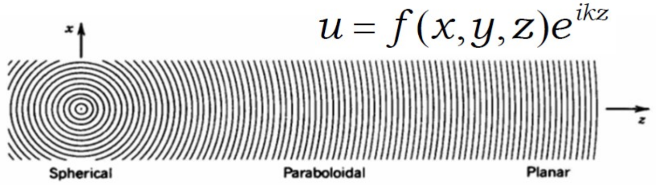
$$\frac{\partial^2 f}{\partial y^2} = f\cdot\left(\frac{i\kappa y}{z}\right)^2 + f\cdot\frac{i\kappa}{z}$$

$$\frac{\partial f}{\partial z} = -\frac{f}{z} + f\cdot\left(\frac{-i\kappa(x^2+y^2)}{2z^2}\right)$$

$$\therefore \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+2ik\frac{\partial f}{\partial z}=0$$

Gaussian beam

$$u = f(x,y,z)e^{ikz}$$



$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + 2ik\frac{\partial f}{\partial z} = 0 \quad \text{(Paraxial Helmholtz Eq.)}$$

Paraboloidal wave

$$f = \frac{A}{z}\exp\left[ik\frac{x^2+y^2}{2z}\right] \xrightarrow{\;z \to z - iz_0\;} f = \frac{A}{z-iz_0}\exp\left[ik\frac{x^2+y^2}{2(z-iz_0)}\right]$$ is also a solution

$$\frac{\partial}{\partial z} = \frac{\partial(z-z_0)}{\partial z}\frac{\partial}{\partial(z-z_0)} = \frac{\partial}{\partial(z-z_0)}$$

Complex beam parameters : $R(z), w(z)$

$$\frac{1}{z-iz_0} = \frac{z+iz_0}{z^2+z_0^2} = \frac{1}{R(z)} + i\frac{2}{kw^2(z)} \longrightarrow R(z) \equiv z\left[1+\left(\frac{z_0}{z}\right)^2\right] \quad w(z) \equiv w_0\left[1+\left(\frac{z}{z_0}\right)^2\right]^{1/2} \quad w_0 \equiv \left(\frac{2z_0}{k}\right)^{1/2}$$

$$f = \frac{A}{z-iz_0}\exp\left[ik\frac{x^2+y^2}{2}\left(\frac{1}{R(z)}+i\frac{2}{kw^2(z)}\right)\right]$$

$$= A_0\frac{w_0}{w(z)}\exp\left[-\frac{\rho^2}{w^2(z)}\right]\exp\left[ik\frac{\rho^2}{2R(z)} - i\varsigma(z)\right]$$

$$\rho \equiv (x^2+y^2)^{1/2}$$

$$A_0 = \frac{A}{z_0}$$

$$\varsigma(z) = -\tan^{-1}\left(\frac{z_0}{z}\right)$$

amplitude

phase

$$\longrightarrow u = fe^{ikz} = A_0\frac{w_0}{w(z)}\exp\left[-\frac{\rho^2}{w^2(z)}\right]\exp\left[ik\left(z+\frac{\rho^2}{2R(z)}\right) - i\varsigma(z)\right]$$

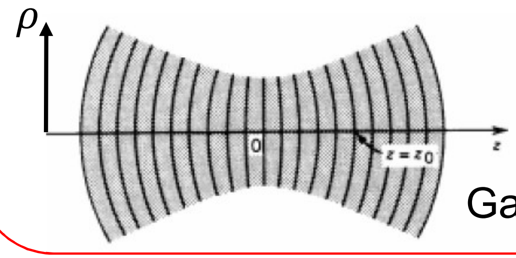
$$\frac{1}{z-iz_0} = Xe^{-i\varsigma(z)} \qquad A_0 = \frac{A}{z_0}$$

$$X = \left|\frac{z+iz_0}{z^2+z_0^2}\right| = \left(\frac{1}{z^2+z_0^2}\right)^{1/2} = \frac{1}{z_0}\left(1+\left(\frac{z}{z_0}\right)^2\right)^{-\frac{1}{2}} = \frac{w_0}{z_0 w(z)}$$

$$e^{-i\ (z)} = \cos(\varsigma(z)) - i\sin(\varsigma(z)) \longrightarrow \tan(\varsigma(z)) = -\frac{z_0}{z}$$

Intensity of the beam

$$I \equiv |u|^2 = I_0\left[\frac{w_0}{w(z)}\right]^2 \exp\left[-\frac{2\rho^2}{w^2(z)}\right]$$



$$I_0 = A_0^2$$

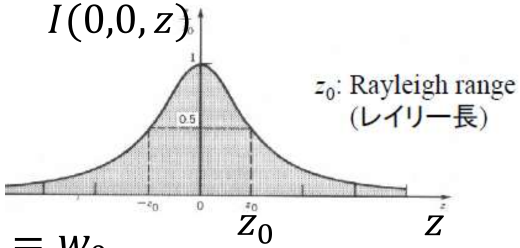
Gaussian beam

Beam parameters

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+2ik\frac{\partial f}{\partial z}=0$$ (Paraxial Helmholtz Eq.)

Rayleigh range z_0 on z-axis $(\rho = 0)$

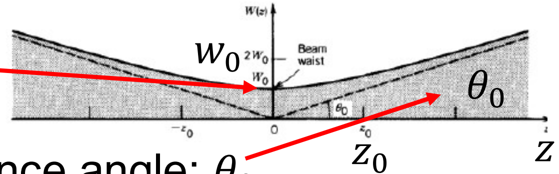
$$I(0,0,z)=I_0\left[\frac{w_0}{w(z)}\right]^2=\frac{I_0}{1+(z/z_0)^2}$$



$$R(z)\equiv z\left[1+\left(\frac{z_0}{z}\right)^2\right] \quad w(z)\equiv w_0\left[1+\left(\frac{z}{z_0}\right)^2\right]^{1/2} \quad w_0\equiv\left(\frac{2z_0}{k}\right)^{1/2}$$

Beam radius: $w(z)$, beam waist: $w(0) = w_0$

$$w(z)\equiv w_0\left[1+\left(\frac{z}{z_0}\right)^2\right]^{1/2} \quad w_0=\left(\frac{\lambda z_0}{\pi}\right)^{1/2}$$



amplitude

Gouy's phase

$$u=fe^{ikz}=A_0\frac{w_0}{w(z)}\exp\left[-\frac{\rho^2}{w^2(z)}\right]\exp\left[ik\left(z+\frac{\rho^2}{2R(z)}\right)-i\varsigma(z)\right]$$

$$\rho\equiv(x^2+y^2)^{1/2}$$

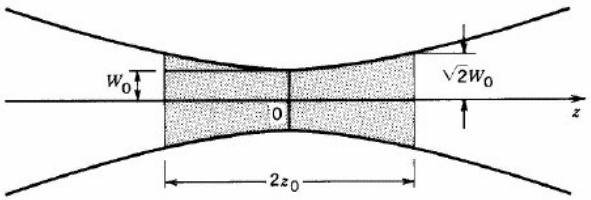
Beam divergence: $w(z)\propto z$, divergence angle: θ_0

$$\theta_0=\lim_{z\to\infty}\frac{w(z)}{z}=\frac{w_0}{z_0}=\frac{\lambda}{\pi w_0}$$

Conforcal parameter: $2z_0$, depth of focus

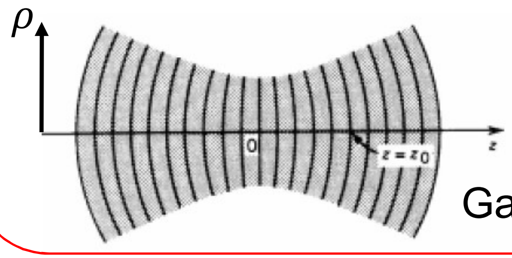
$$I(0,0,z_0)=\frac{I(0,0,0)}{2}$$

$$w(z_0)=\frac{w_0}{2}$$



Intensity of the beam

$$I\equiv|u|^2=I_0\left[\frac{w_0}{w(z)}\right]^2\exp\left[-\frac{2\rho^2}{w^2(z)}\right]$$



$$I_0 = A_0^2$$

Gaussian beam

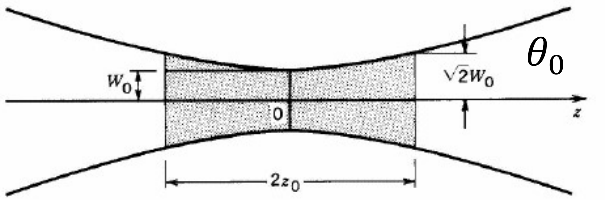
Beam can not be focused < $2w_0$

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+2ik\frac{\partial f}{\partial z}=0$$ (Paraxial Helmholtz Eq.)

Conforcal parameter: $2z_0$, depth of focus

$$I(0,0,z_0)=\frac{I(0,0,0)}{2}$$

$$w(z_0)=\sqrt{2}w_0$$

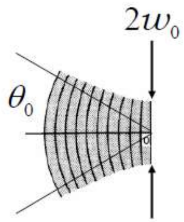


$$R(z)\equiv z\left[1+\left(\frac{z_0}{z}\right)^2\right] \quad w(z)\equiv w_0\left[1+\left(\frac{z}{z_0}\right)^2\right]^{1/2} \quad w_0\equiv\left(\frac{2z_0}{k}\right)^{1/2}$$

amplitude   Gouy's phase

$$u=fe^{ikz}=A_0\frac{w_0}{w(z)}\exp\left[-\frac{\rho^2}{w^2(z)}\right]\exp\left[ik\left(z+\frac{\rho^2}{2R(z)}\right)-i\varsigma(z)\right]$$

$$\rho\equiv(x^2+y^2)^{1/2}$$



Uncertainty principles: $\Delta z\Delta p_z>\hbar/2$

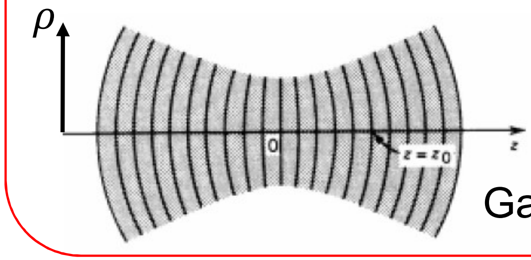
$$\Delta z\sim w_0,\qquad \Delta p_z\sim\frac{h}{\Delta z}=\frac{h}{w_0}\qquad\qquad p_z=\frac{h}{\lambda}$$

$$\theta\equiv\frac{\Delta p_z}{p_z}=\frac{h}{w_0}\cdot\frac{\lambda}{h}=\frac{\lambda}{w_0}\longrightarrow w_0\sim\frac{\lambda}{\theta}<\lambda\cdot\frac{2}{\pi}<\lambda,\ (\theta<\frac{\pi}{2})$$

We can not focus less than λ!

Intensity of the beam

$$I\equiv|u|^2=I_0\left[\frac{w_0}{w(z)}\right]^2\exp\left[-\frac{2\rho^2}{w^2(z)}\right]$$



$I_0=A_0^2$

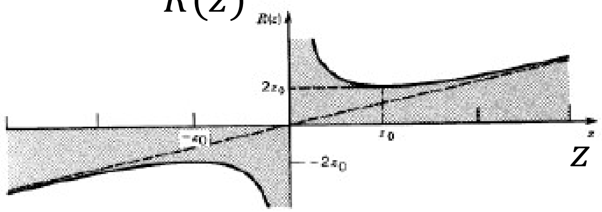
Gaussian beam

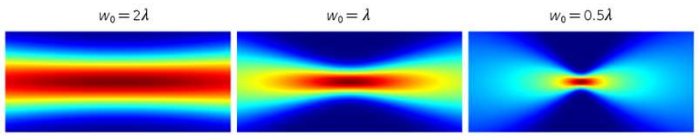
Radius of wavefront: $R(z)$

$$R(z)\equiv z\left[1+\left(\frac{z_0}{z}\right)^2\right]\xrightarrow{z>>z_0} z$$

$$u\propto\exp\left[ik\left(z+\frac{\rho^2}{2z}\right)\right]$$

$R(z)$





Higher order Gaussian Beam

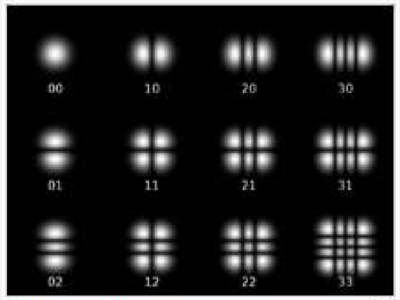
Hermite-Gaussian mode

$$u = f(x,y,z)e^{ikz}$$

$$u_J(x,z) = \left(\frac{\sqrt{2/\pi}}{2^J\ J!\ w_0}\right)^{1/2}\left(\frac{q_0}{q(z)}\right)^{1/2}\left(-\frac{q^*(z)}{q(z)}\right)^{J/2} H_J\left(\frac{\sqrt{2}x}{w(z)}\right)\ \exp\left(-i\frac{kx^2}{2q(z)}\right)$$

Lindlein N., Leuchs G. (2007) Wave Optics. pp152-155 In: Träger F. (eds)

Springer Handbook of Lasers and Optics. Springer Handbooks. Springer, New York, NY



Twelve Hermite-Gaussian modes

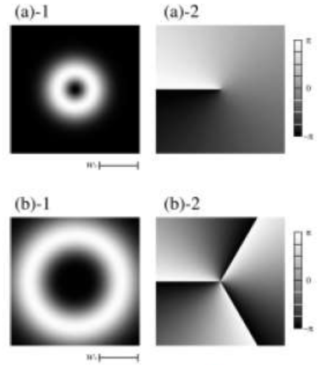


Fig. 1 Intensity distributions ((a)-1, (b)-1) and phase distributions ((a)-2, (b)-2) of Laguerre-Gaussian beams at beam waist. (a) $p = 0$, $m = 1$. (b) $p = 0$, $m = 3$.

Laguerre-Gaussian mode

$$u(r,\phi,z) = C_{lp}^{LG}\frac{w_0}{w(z)}\left(\frac{r\sqrt{2}}{w(z)}\right)^{|l|}\exp\left(-\frac{r^2}{w^2(z)}\right)L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\times$$
$$\exp\left(-ik\frac{r^2}{2R(z)}\right)\exp(-il\phi)\ \exp(i\psi(z)) \ .$$

$$C_{lp}^{LG} = \sqrt{\frac{2p!}{\pi(p+|l|)!}} \Rightarrow \int_0^{2\pi}d\phi\int_0^{\infty}rdr|u(r,\phi,z)|^2 = 1.$$

H. Kogelnik, T. Li, Proc. IEEE 54, 1312 (1966)

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} - 2jk\frac{\partial\psi}{\partial z} = 0 \qquad (11)$$

b) Modes in Cylindrical Coordinates: For a system with a cylindrical (r, ϕ, z) geometry one uses a trial solution for (11) of the form

$$\psi = g\left(\frac{r}{w}\right)\cdot\exp\left\{-j\left(P + \frac{k}{2q}r^2 + l\phi\right)\right\}. \qquad (35)$$

After some calculation one finds

$$g = \left(\sqrt{2}\ \frac{r}{w}\right)^l\cdot L_p{}^l\left(2\ \frac{r^2}{w^2}\right) \qquad (36)$$

where $L_p{}^l$ is a generalized Laguerre polynomial, and p and l are the radial and angular mode numbers. $L_p{}^l(x)$ obeys the differential equation

$$x\frac{d^2L_p{}^l}{dx^2} + (l + 1 - x)\frac{dL_p{}^l}{dx} + pL_p{}^l = 0. \qquad (37)$$

Since I could not find the derivation of Laguerre-Gaussian mode,
hereafter I solved by myself!!

Laguerre-Gaussian mode

H. Kogelnik, T. Li. Proc. IEEE 54, 1312 (1966)

Paraxial Helmholtz Eq.

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+2i\kappa\frac{\partial\Psi}{\partial z}=0$$

Cylindrical coordinate

$$\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$$

We assume the shape of Ψ

$$\Psi=g\left(\frac{r}{w(z)}\right)\exp\left\{-iP(z)+\frac{i\kappa}{2q(z)}r^2-i\ell\varphi\right\}\equiv gF(r,\varphi,z)$$

$$\frac{\partial\Psi}{\partial r}=\overset{\text{A}}{\frac{1}{w}g'F}+\overset{\text{B}}{\frac{i\kappa r}{q}gF}$$

$$\frac{\partial^2\Psi}{\partial r^2}=\overset{\text{C}}{\frac{1}{w^2}g''F}+\overset{\text{D}}{\frac{2i\kappa r}{wq}g'F}-\overset{\text{E}}{\frac{\kappa^2r^2}{q^2}gF}+\overset{\text{B'}}{\frac{i\kappa}{q}gF}$$

$$\frac{\partial^2\Psi}{\partial\varphi^2}=\overset{\text{F}}{-\ell^2gF}$$

$$\frac{\partial\Psi}{\partial z}=\overset{\text{G}}{-\frac{rw'}{w^2}g'F}+\left(\overset{\text{H}}{-iP'}-\overset{\text{I}}{\frac{i\kappa q'}{2q^2}r^2}\right)gF$$

$$\frac{1}{gF}\times\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}+2i\kappa\frac{\partial}{\partial z}\right)\Psi$$

$$=\overset{\text{C}}{\frac{1}{w^2}\frac{g''}{g}}+\left(\overset{\text{A}}{\frac{1}{wr}}-\overset{\text{G}}{\frac{2i\kappa rw'}{w^2}}+\overset{\text{D}}{\frac{2i\kappa r}{wq}}\right)\frac{g'}{g}+\overset{\text{BB'}}{\frac{2i\kappa}{q}}-\overset{\text{F}}{\frac{\ell^2}{r^2}}+\overset{\text{H}}{2\kappa P'}+\underbrace{\overset{\text{I}}{\frac{\kappa^2q'r^2}{q^2}}-\overset{\text{E}}{\frac{\kappa^2r^2}{q^2}}}_{=0}=0$$

P. Helmholtz Eq. should not diverge for $r\to\infty$

$$\frac{\kappa^2r^2}{q^2}(q'-1)r^2\propto r^2$$

$$\to\quad q'=1\ (r\to\infty)\to q=z-iz_0$$

Variable separation

$$\frac{\partial \Psi}{\partial x^2}+\frac{\partial \Psi}{\partial y^2}+2i\kappa\frac{\partial \Psi}{\partial z}=0 \qquad \Psi = g\left(\frac{r}{w(z)}\right)\exp\left\{-iP(z)+\frac{i\kappa}{2q(z)}r^2 - i\ell\varphi\right\} \equiv gF(r,\varphi,z)$$

$$\frac{1}{gF}\times\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \varphi^2}+2i\kappa\frac{\partial}{\partial z}\right)\Psi$$

$$=\underbrace{\frac{1}{w^2}\frac{g''}{g}}_{C}+\left(\underbrace{\frac{1}{wr}}_{A}-\underbrace{\frac{2i\kappa r w'}{w^2}}_{G}+\underbrace{\frac{2i\kappa r}{wq}}_{D}\right)\frac{g'}{g}+\underbrace{\frac{2i\kappa}{q}}_{BB'}-\underbrace{\frac{\ell^2}{r^2}}_{F}+\underbrace{2\kappa P'}_{H}+\underbrace{\frac{\kappa^2 q' r^2}{q^2}}_{I}-\underbrace{\frac{\kappa^2 r^2}{q^2}}_{E}=0$$

$(q'=1)$, $I - E = 0$

$$\underbrace{\frac{1}{w^2}\frac{g''}{g}}_{C}+\left(\underbrace{\frac{1}{wr}}_{A}-\underbrace{\frac{4r}{w^3}}_{GD}\right)\frac{g'}{g}+\underbrace{\frac{2i\kappa}{q}}_{BB'}-\underbrace{\frac{\ell^2}{r^2}}_{F}-\underbrace{2\kappa P'}_{H}=0$$

$w(z), q(z), P(z)$ are functions of z.

Variable separation

$$\begin{cases}\underbrace{\frac{2i\kappa}{q}}_{BB'}+\underbrace{2\kappa P'}_{H}\equiv f(z)\\ \underbrace{\frac{1}{w^2}\frac{g''}{g}}_{C}+\left(\underbrace{\frac{1}{wr}}_{A}-\underbrace{\frac{4r}{w^3}}_{GD}\right)\frac{g'}{g}-\underbrace{\frac{\ell^2}{r^2}}_{F}=-f(z)\end{cases}$$

$\times w^2 g \qquad \alpha \equiv w^2 f(z)$

$$\therefore\ g''+\left(\frac{w}{r}-\frac{4r}{w}\right)g'-\left(\frac{w^2\ell^2}{r^2}-\alpha\right)g=0$$

Analogous to Gaussian beam

$$\frac{1}{q}=\frac{1}{R}+\frac{i\lambda}{\pi w^2},$$

square $\rightarrow \left(\frac{1}{q}\right)^2=\frac{1}{R^2}-\frac{\lambda^2}{\pi^2 w^4}+\frac{2i\lambda}{\pi R w^2}$

derivatieve $\rightarrow -\left(\frac{1}{q}\right)'=\frac{q'}{q^2}=\frac{1}{q^2}=\frac{R'}{R^2}+\frac{2i\lambda w'}{\pi w^3}$

$(q'=1)$

Compare real and imaginary parts

$$\therefore R'=1-\frac{\lambda^2 R^2}{\pi^2 w^4},\quad w'=\frac{w}{R}$$

$$\underbrace{\frac{1}{wr}}_{A}-\underbrace{\frac{2i\kappa r w'}{w^2}}_{G}+\underbrace{\frac{2i\kappa r}{wq}}_{D}=\frac{1}{wr}-\frac{2i\kappa r}{wR}+\frac{2i\kappa r}{w}\left(\frac{1}{R}+\frac{i\lambda}{\pi w^2}\right)$$

$$=\frac{1}{wr}-\frac{2\kappa r\lambda}{\pi w^3}$$

$$=\frac{1}{wr}-\frac{4r}{w^3}\qquad(\kappa\lambda=2\pi)$$

Laguerre differential Eq.

$$\frac{\partial \Psi}{\partial x^2}+\frac{\partial \Psi}{\partial y^2}+2i\kappa\frac{\partial \Psi}{\partial z}=0 \qquad \Psi=g\left(\frac{r}{w(z)}\right)\exp\left\{-iP(z)+\frac{i\kappa}{2q(z)}r^2-i\ell\varphi\right\}\equiv gF(r,\varphi,z)$$

$$\therefore\ g''+\left(\frac{w}{r}-\frac{4r}{w}\right)g'-\left(\frac{w^2\ell^2}{r^2}-\alpha\right)g=0$$

$$x=\frac{2r^2}{w^2},\qquad \frac{r}{w}=\sqrt{\frac{x}{2}},\qquad \frac{w}{r}=\sqrt{\frac{2}{x}}$$

$$\frac{dx}{d\left(\frac{r}{w}\right)}=\frac{4r}{w}=4\sqrt{\frac{x}{2}}=2\sqrt{2x}$$

$$\frac{d\sqrt{x}}{d\left(\frac{r}{w}\right)}=\frac{d\sqrt{x}}{dx}\frac{dx}{d\left(\frac{r}{w}\right)}=\frac{1}{2\sqrt{x}}2\sqrt{2x}=\sqrt{2}$$

$$g'=\frac{dg}{d\left(\frac{r}{w}\right)}=\frac{dg}{dx}\frac{dx}{d\left(\frac{r}{w}\right)}=2\sqrt{2x}\frac{dg}{dx}$$

$$g''=8x\frac{d^2g}{dx^2}+4\frac{dg}{dx}$$

We assume the shape of g $\qquad g\left(\frac{r}{w(z)}\right)=x^{\frac{\ell}{2}}L(x)$

$$8x\frac{d^2g}{dx^2}+4\frac{dg}{dx}+\left(\sqrt{\frac{2}{x}}-2\sqrt{2x}\right)2\sqrt{2x}\frac{dg}{dx}-\left(\frac{2\ell^2}{x}-\alpha\right)g=0$$

$$8x\frac{d^2g}{dx^2}+(8-8x)\frac{dg}{dx}-\left(\frac{2\ell^2}{x}-\alpha\right)g=0$$

$$\frac{dg}{dx}=x^{\frac{\ell}{2}}L'+\frac{\ell}{2}x^{\frac{\ell}{2}-1}L,$$

$$\frac{d^2g}{dx^2}=x^{\frac{\ell}{2}}L''+\ell x^{\frac{\ell}{2}-1}L'+\frac{\ell}{2}\left(\frac{\ell}{2}-1\right)x^{\frac{\ell}{2}-2}L$$

$$8x^{\frac{\ell}{2}+1}L''+8\ell x^{\frac{\ell}{2}}L'+2\ell(\ell-2)x^{\frac{\ell}{2}-1}L$$
$$+(8-8x)\left(x^{\frac{\ell}{2}}L'+\frac{\ell}{2}x^{\frac{\ell}{2}-1}L\right)-\left(\frac{2\ell^2}{x}-\alpha\right)x^{\frac{\ell}{2}}L=0$$

$\div\ 8x^{\frac{\ell}{2}}$

$$xL''+\ell L'+\ell(\ell-2)\frac{1}{4x}L+(1-x)\left(L'+\frac{\ell}{2x}L\right)-\left(\frac{\ell^2}{4x}-\frac{\alpha}{8}\right)L=0$$

$$xL''+(\ell+1-x)L'-\left(\frac{\ell}{2}-\frac{\alpha}{8}\right)L=0$$

$$-\left(\frac{\ell}{2}-\frac{\alpha}{8}\right)\equiv n$$

$$xL''+(\ell+1-x)L'+nL=0$$

Laguerre differential Eq.

$$\frac{\partial \Psi}{\partial x^2} + \frac{\partial \Psi}{\partial y^2} + 2i\kappa \frac{\partial \Psi}{\partial z} = 0$$

Step1

$$\Psi = g\left(\frac{r}{w(z)}\right) \exp\left\{-iP(z) + \frac{i\kappa}{2q(z)} r^2 - i\ell\varphi\right\}$$

Step2 $q' = 1 \rightarrow q(z) = z + q_0 \equiv z - iz_0$

Step3 $g\left(\frac{r}{w(z)}\right) = x^{\frac{\ell}{2}} L_n^\ell(x) \qquad x = \frac{2r^2}{w^2}$

Laguerre differential Eq.

$$x L_n^{\ell\,\prime\prime} + (\ell + 1 - x) L_n^{\ell\,\prime} + n L_n^\ell = 0$$

$n \equiv -\left(\frac{\ell}{2} - \frac{\alpha}{8}\right) \equiv$: integer$\rightarrow L = L_n^\ell(x)$

$$\alpha \equiv w^2 f(z) = 8n + 4\ell$$

H. Kogelnik, T. Li, Proc. IEEE 54, 1312 (1966)

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} - 2jk \frac{\partial \psi}{\partial z} = 0 \qquad (11)$$

b) Modes in Cylindrical Coordinates: For a system with a cylindrical (r, ϕ, z) geometry one uses a trial solution for (11) of the form

$$\psi = g\left(\frac{r}{w}\right) \cdot \exp\left\{-j\left(P + \frac{k}{2q} r^2 + l\phi\right)\right\}. \qquad (35)$$

After some calculation one finds

$$g = \left(\sqrt{2}\,\frac{r}{w}\right)^l \cdot L_p^l\left(2\,\frac{r^2}{w^2}\right) \qquad (36)$$

where L_p^l is a generalized Laguerre polynomial, and p and l are the radial and angular mode numbers. $L_p^l(x)$ obeys the differential equation

$$x \frac{d^2 L_p^l}{dx^2} + (l + 1 - x) \frac{dL_p^l}{dx} + pL_p^l = 0. \qquad (37)$$

Associated Laguerre polynomials: L_n^ℓ

$$L_0^\ell(x) = 1$$

$$L_1^\ell(x) = 1 + \ell - x$$

Recursion formula

$$L_{n+1}^\ell(x) = \frac{(2n + 1 + \ell - x) L_n^\ell(x) - (n + \ell) L_{n-1}^\ell(x)}{n + 1}$$

Rodrigues's Formula

$$L_n^\ell(x) = \frac{x^{-\ell} e^x}{n!} \frac{d^n}{dx^n} \left(e^{-x} x^{n+\ell}\right)$$

Generating function

$$\sum_{n=0}^{\infty} t^n L_n^\ell(x) = \frac{e^{-\frac{tx}{1-t}}}{(1-t)^{\ell+1}}$$

$\ell = 0$ Laguerre polynomials $L_n^0 = L_n$

$L_0(x) = 1$

$L_1(x) = -x + 1$

$L_2(x) = x^2 - 4x + 2$

$L_3(x) = -x^3 + 9x^2 - 18x + 6$

Recursion formula

$$x \frac{d}{dx} L_n(x) = n L_n(x) - n^2 L_{n-1}(x)$$

$$L_{n+1}(x) = (2n + 1 - x) L_n(x) - n^2 L_{n-1}(x)$$

Compare with Gaussian beam

$$xL_n^{\ell\,''} + (\ell + 1 - x)L_n^{\ell\,'} + nL_n^{\ell} = 0$$

$$\frac{\partial \Psi}{\partial x^2} + \frac{\partial \Psi}{\partial y^2} + 2i\kappa \frac{\partial \Psi}{\partial z} = 0$$

$$n \equiv -\left(\frac{\ell}{2} - \frac{\alpha}{8}\right) : \text{integer}$$

$$\alpha \equiv w^2 f(z) = 8n + 4\ell$$

$$\Psi = g\left(\frac{r}{w(z)}\right)\exp\left\{-iP(z) + \frac{i\kappa}{2q(z)}r^2 - i\ell\varphi\right\} \equiv gF(r,\varphi,z)$$

$$g\left(\frac{r}{w(z)}\right) = x^{\frac{\ell}{2}}L_n^{\ell}(x) \qquad q' = 1 \rightarrow q \equiv z - iz_0$$

BB' H $\div 2i$

$$\frac{2i\kappa}{q} + 2\kappa P' \equiv f(z) = \frac{8n + 4\ell}{w^2} \rightarrow \frac{1}{q} = iP' - \frac{(4n + 2\ell)i}{w^2}$$

$$\frac{1}{q} = \frac{1}{R} + \frac{2i}{\kappa w^2} = iP' - \frac{i(4n + 2\ell)}{\kappa w^2}$$

$$iP' = \frac{1}{R} + \frac{2i}{\kappa w^2} + \frac{i(4n + 2\ell)}{\kappa w^2} = \frac{1}{R} + \frac{i(4n + 2\ell + 2)}{\kappa w^2}$$

$$P' = \frac{1}{iR} + \frac{(4n + 2\ell + 2)}{\kappa w^2} = \frac{z}{i(z^2 + z_0^2)} + \frac{z_0(2n + \ell + 1)}{(z^2 + z_0^2)}$$

$$P = \int_0^z P' dz = \frac{\log \dfrac{z^2 + z_0^2}{z_0^2}}{2i} + (2n + \ell + 1)\psi(z)$$

$$\exp\{-iP(z)\} = \left(1 + \frac{z^2}{z_0^2}\right)^{-\frac{1}{2}} \times \exp\{-i(2n + \ell + 1)\psi(z)\}$$

Gaussian beam

$$\frac{1}{q(z)} = \frac{1}{z - iz_0} = \frac{z + iz_0}{z^2 + z_0^2} = \frac{1}{R(z)} + i\frac{2}{kw^2(z)}$$

$$\frac{1}{R(z)} = \frac{z}{z^2 + z_0^2}, \qquad \frac{2}{\kappa w^2(z)} = \frac{z_0}{z^2 + z_0^2},$$

$$f = \frac{A}{z - iz_0}\exp\left[ik\frac{x^2 + y^2}{2(z - iz_0)}\right] \qquad A_0 = \frac{A}{z_0}$$

$$f = \frac{A}{z - iz_0}\exp\left[ik\frac{x^2 + y^2}{2}\left(\frac{1}{R(z)} + i\frac{2}{kw^2(z)}\right)\right]$$

$$= A_0\frac{w_0}{w(z)}\exp\left[-\frac{\rho^2}{w^2(z)}\right]\exp\left[ik\frac{\rho^2}{2R(z)} - i\varsigma(z)\right]$$

$$\int_0^z dz \frac{z}{(z^2 + z_0^2)} = \frac{1}{2}\log\frac{z^2 + z_0^2}{z_0^2} \qquad \tan(\varsigma(z)) = -\frac{z_0}{z}$$

$$\int_0^z dz \frac{z_0}{(z^2 + z_0^2)} = \arctan\left(\frac{z}{z_0}\right) \equiv \psi(z) = \frac{\pi}{2} + \varsigma(z)$$

Solution of Laguerre Gauss mode

Paraxial Helmholtz Eq.

$$\frac{\partial \Psi}{\partial x^2} + \frac{\partial \Psi}{\partial y^2} + 2i\kappa \frac{\partial \Psi}{\partial z} = 0 \qquad -\left(\frac{\ell}{2} - \frac{\alpha}{8}\right) \equiv n\text{: integer}$$

$$\Psi = g\left(\frac{r}{w(z)}\right) \exp\left\{-iP(z) + \frac{i\kappa}{2q(z)} r^2 - i\ell\varphi\right\}$$

$$g\left(\frac{r}{w(z)}\right) = x^{\frac{\ell}{2}} L_n^{\ell}(x) = \left(\frac{\sqrt{2}\, r}{w(z)}\right)^{\ell} L_n^{\ell}\left(\frac{2r^2}{w^2(z)}\right) \qquad x = \frac{2r^2}{w^2}$$

$$\exp\{-iP(z)\} = \left(1 + \frac{z^2}{z_0^2}\right)^{-\frac{1}{2}} \times \exp\{-i(2n + \ell + 1)\psi(z)\}$$

$$\exp\left\{\frac{i\kappa}{2q(z)} r^2\right\} = \exp\left(-\frac{r^2}{w^2(z)}\right) \exp\left(i\kappa \frac{r^2}{2R(z)}\right)$$

$$\Psi = \frac{w_0}{w(z)} \left(\frac{\sqrt{2}\, r}{w(z)}\right)^{\ell} L_n^{\ell}\left(\frac{2r^2}{w^2(z)}\right) \exp\left(-\frac{r^2}{w^2(z)}\right) \exp\left(i\kappa \frac{r^2}{2R(z)} - i\ell\varphi - i(2n + \ell + 1)\psi(z)\right)$$

$$x L_n^{\ell\,\prime\prime}(x) + (\ell + 1 - x) L_n^{\ell\,\prime}(x) + n L_n^{\ell}(x) = 0 \qquad x = \frac{2r^2}{w^2}$$

$$R(z) = z\left\{1 + \left(\frac{z_0}{z}\right)^2\right\}, \quad w(z) = w_0\left\{1 + \left(\frac{z}{z_0}\right)^2\right\}^{\frac{1}{2}}, \quad w_0 \equiv \left(\frac{2z_0}{\kappa}\right)^{\frac{1}{2}}, \quad \psi(z) \equiv \arctan\left(\frac{z}{z_0}\right)$$

Gaussian beam

$$\frac{1}{q(z)} = \frac{1}{z - iz_0} = \frac{z + iz_0}{z^2 + z_0^2} = \frac{1}{R(z)} + i\frac{2}{kw^2(z)}$$

$$R(z) = z\left\{1 + \left(\frac{z_0}{z}\right)^2\right\}, \quad w(z) = w_0\left\{1 + \left(\frac{z}{z_0}\right)^2\right\}^{\frac{1}{2}}, \quad w_0 \equiv \left(\frac{2z_0}{\kappa}\right)^{\frac{1}{2}}$$

$$\left(1 + \frac{z^2}{z_0^2}\right)^{-\frac{1}{2}} = \frac{w_0}{w(z)}$$

$$f = \frac{A}{z - iz_0} \exp\left[ik\frac{x^2 + y^2}{2(z - iz_0)}\right] \qquad A_0 = \frac{A}{z_0}$$

$$f = \frac{A}{z - iz_0} \exp\left[ik\frac{x^2 + y^2}{2}\left(\frac{1}{R(z)} + i\frac{2}{kw^2(z)}\right)\right]$$

$$= A_0 \frac{w_0}{w(z)} \exp\left[-\frac{\rho^2}{w^2(z)}\right] \exp\left[ik\frac{\rho^2}{2R(z)} - i\varsigma(z)\right]$$

Comparison with previous solutions for Laguerre Gauss modes

Present results $\frac{\partial\Psi}{\partial x^2}+\frac{\partial\Psi}{\partial y^2}+2i\kappa\frac{\partial\Psi}{\partial z}=0$ $\quad -\left(\frac{\ell}{2}-\frac{\alpha}{8}\right)\equiv n$: integer

$$\Psi=\frac{w_0}{w(z)}\left(\frac{\sqrt{2}\,r}{w(z)}\right)^{\ell}L_n^{\ell}\left(\frac{2r^2}{w^2(z)}\right)\exp\left(-\frac{r^2}{w^2(z)}\right)\exp\left(i\kappa\frac{r^2}{2R(z)}-i\ell\varphi-i(2n+\ell+1)\psi(z)\right)$$

$$xL_n^{\ell\,\prime\prime}(x)+(\ell+1-x)L_n^{\ell\,\prime}(x)+nL_n^{\ell}(x)=0 \qquad x=\frac{2r^2}{w^2}$$

$$R(z)=z\left\{1+\left(\frac{z_0}{z}\right)^2\right\},\quad w(z)=w_0\left\{1+\left(\frac{z}{z_0}\right)^2\right\}^{\frac{1}{2}},\quad w_0\equiv\left(\frac{2z_0}{\kappa}\right)^{\frac{1}{2}},\quad \psi(z)\equiv\arctan\left(\frac{z}{z_0}\right)$$

Gaussian beam $n=\ell=0, L_0^0(x)=1$

$$\frac{1}{q(z)}=\frac{1}{z-iz_0}=\frac{z+iz_0}{z^2+z_0^2}=\frac{1}{R(z)}+i\frac{2}{kw^2(z)}$$

$$R(z)=z\left\{1+\left(\frac{z_0}{z}\right)^2\right\},\quad w(z)=w_0\left\{1+\left(\frac{z}{z_0}\right)^2\right\}^{\frac{1}{2}},\quad w_0\equiv\left(\frac{2z_0}{\kappa}\right)^{\frac{1}{2}}$$

$$\left(1+\frac{z^2}{z_0^2}\right)^{-\frac{1}{2}}=\frac{w_0}{w(z)}$$

$$f=\frac{A}{z-iz_0}\exp\left[ik\frac{x^2+y^2}{2(z-iz_0)}\right]$$

$$A_0=\frac{A}{z_0}$$

$$f=\frac{A}{z-iz_0}\exp\left[ik\frac{x^2+y^2}{2}\left(\frac{1}{R(z)}+i\frac{2}{kw^2(z)}\right)\right]$$

$$=A_0\frac{w_0}{w(z)}\exp\left[-\frac{\rho^2}{w^2(z)}\right]\exp\left[ik\frac{\rho^2}{2R(z)}-i\varsigma(z)\right]$$

H. Kogelnik, T. Li, Proc. IEEE 54, 1312 (1966)

$$\frac{\partial\Psi}{\partial x^2}+\frac{\partial\Psi}{\partial y^2}-2i\kappa\frac{\partial\Psi}{\partial z}=0$$

$$u(r,\phi,z)=C_{lp}^{LG}\frac{w_0}{w(z)}\left(\frac{r\sqrt{2}}{w(z)}\right)^{|l|}\exp\left(-\frac{r^2}{w^2(z)}\right)L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\times$$
$$\exp\left(-ik\frac{r^2}{2R(z)}\right)\exp(-il\phi)\exp(i\psi(z))\ .$$

$$C_{lp}^{LG}=\sqrt{\frac{2p!}{\pi(p+|l|)!}}\Rightarrow\int_0^{2\pi}d\phi\int_0^{\infty}rdr|u(r,\phi,z)|^2=1.$$

W. Paufler et al, J. Opt 21 094001 (2019)

$$LG_{l,p}(\rho,\varphi,z):=u(\mathbf{r})=E_0\frac{W_0}{W(z)}\left(\frac{\sqrt{2}\rho}{W(z)}\right)^{|l|}L_p^{|l|}\left[\frac{2\rho^2}{W^2(z)}\right]$$
$$\times\exp\left(-\frac{\rho^2}{W^2(z)}\right)\exp\left(ik\frac{\rho^2}{2R(z)}+i\Phi_G(z)+il\varphi\right). \qquad (6)$$

Here W_0 is the beam waist, $W(z)=W_0\sqrt{1+\frac{z^2}{z_r^2}}$ is the beam width and $z_r=k\frac{W_0^2}{2}$ is the Rayleigh range. The Gouy phase $\Phi_G(z)=-(|\ell|+2p+1)\arctan\left(\frac{z}{z_r}\right)$ refers to a phase shift when passing through the focus of the beam, $L_p^{|\ell|}[x]$ are the associated Laguerre polynomials and $R(z)=z\left(1+\frac{z_r^2}{z^2}\right)$ is the phase front radius. The OAM ℓ induces the azimuthal

Lindlein N., Leuchs G. (2007) Wave Optics. In: Träger F. (eds) Springer Handbook of Lasers and Optics. Springer Handbooks. Springer, NY

$$u_{p,l}^{LG}(r,\phi,z)=C_{p,l}^{LG}\frac{1}{w(z)}\left(\frac{\sqrt{2}r}{w(z)}\right)^{|l|}$$
$$L_p^{|l|}\left(\frac{2r^2}{w^2(z)}\right)\exp(-il\phi)$$
$$\times\exp\left[-\frac{r^2}{w^2(z)}\right]\exp\left[-\frac{ikr^2}{2R(z)}\right]$$
$$\exp[i(2p+|l|+1)\psi(z)] \quad ($$